# Two-species $d$-dimensional diffusive model and its mapping onto a growth model

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In this work, we consider a diffusive two-species $d$-dimensional model and study it in great detail. Two types of particles, with hard core, diffuse symmetrically and cross each other. For arbitrary dimensions, we obtain the exact density, the instantaneous, as well as noninstantaneous, two-point correlation functions for various initial conditions. We study the impact of correlations in the initial state on the dynamics. Finally, we map the one-dimensional version of the model under consideration onto a restricted solid-on-solid growth model with three states and solve its dynamics.

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## I. INTRODUCTION

Stochastic reaction-diffusion models have recently attracted much interest in the last decade (see, e.g., [1,2] and references therein); the latter appeared (directly or via mapping) as models for traffic flow [3], kinetic biopolymerization, [4], reptation of DNA in gels [5], interface growth [6,7], etc.

In this context, simple symmetric (SEP) [8] and asymmetric exclusion processes, in one dimension (ASEP) [1,2,9] play a particular role because of their relationships with integrable quantum spin systems (Heisenberg chains) and because of their connection with the Kardar-Parisi-Zhang (KPZ) equation [10], directed polymers in random media [6], and shock formation (see, e.g., [1] and references therein). These models have been extensively studied and the ASEP with open boundary conditions, as a simple driven diffusion model, exhibits a rich dynamical behavior involving different nonequilibrium phase transitions in the steady states. They can be studied exactly on the basis of the so-called *matrix approach* (MA), an algebraic approach based on an ansatz for the probability distribution which is related to the integrability of some quantum spin chains. This approach provides the full solution of the ASEP (and also the SEP) model, including the full phase diagram, density profile and, in principle, any equal-time correlation functions. Though, only few explicit results are known [11] about the dynamical correlation functions, much work has been done on the static properties.

The MA has been generalized to solve the stationary states of one-dimensional models with several species [12] and, recently, a first-order phase transition in some models has been found [13] (see also [14] where different results were obtained, independently, for the same model).

The lack of exact results for the dynamics of multispecies models [15,16] (in particular in dimension $d>1$ see also [17] and references therein), has motivated us to study in some detail the *dynamics* of a two-species model, which is related to the models introduced by Arndt *et al.* in [12,13]. We compute explicitly, in arbitrary dimensions, the density, and the two-point instantaneous and noninstantaneous correlation functions. We then exploit the exact results for the two-point correlation function to study a restricted solid-on-solid (RSOS) [6,7,18,19] growth model.

The paper is organized as follows. In Sec. II, we present the general *stochastic* formalism within which we will work. In Sec. III, for the model under consideration, we compute in arbitrary dimensions, the density for various initial states and in the presence or absence of initial correlations. In Sec. IV, for a translationally invariant version of our model, we evaluate the instantaneous two-point correlation functions in arbitrary dimensions. In particular, in one-spatial dimension, we assume both cases where initial correlations are absent or present. In Sec. V, we introduce and solve a growth model of RSOS type "with three states." This analysis is carried out for different initial states (correlated and uncorrelated). Finally, in Sec. VI, we calculate for systems with uncorrelated (but random) as well as correlated initial states the noninstantaneous two-point correlation functions.

## II. THE FORMALISM AND THE MODEL

Consider an hypercubic lattice of dimension $d$ with $N$ sites ($N=L^d$), where $L$ represents the linear dimension of the hypercube, and periodic boundary conditions are imposed. Further, assume that local bimolecular reactions between species $A$ and $B$ takes place. Each site is either empty (denoted by the symbol 0) or occupied at most by one particle of type $A$ (respectively, $B$) denoted in the following by the index 1 (respectively, 2). The dynamics are parametrized by the transition rates $\Gamma_{\alpha\beta}^{\gamma\delta}$, where $\alpha,\beta,\gamma,\delta=0,1,2$: $\forall(\alpha,\beta)\neq(\gamma,\delta)$, $\Gamma_{\alpha\beta}^{\gamma\delta}$: $\alpha+\beta\rightarrow\gamma+\delta$.

Probability conservation implies $\Gamma_{\alpha\beta}^{\alpha\beta}=-\Sigma_{(\alpha,\beta)\neq(\alpha',\beta')}\times\Gamma_{\alpha\beta}^{\alpha'\beta'}$, with $\Gamma_{\alpha\beta}^{\gamma\delta}\geqslant0$, $\forall(\alpha,\beta)\neq(\gamma,\delta)$.

For example, the rate $\Gamma_{22}^{12}$ corresponds to the process $BB\rightarrow AB$, while conservation of probability leads to $\Gamma_{11}^{11}=-(\Gamma_{11}^{10}+\Gamma_{11}^{01}+\Gamma_{11}^{00}+\Gamma_{11}^{02}+\Gamma_{11}^{20}+\Gamma_{11}^{21}+\Gamma_{11}^{12}+\Gamma_{11}^{22})$.

The state of the system is determined by specifying the probability for the occurence of configuration $\{n\}$ at time $t$. It is represented by the ket $|P(t)\rangle=\Sigma_{\{n\}}P(\{n\},t)|\{n\}\rangle$, where the sum runs over the $3^N$ configurations ($N=L^d$). At site $i$ the local state is denoted by the ket $|n_i\rangle=(100)^T$ if the site $i$ is empty, $|n_i\rangle=(010)^T$ if the site $i$ is occupied by a particle of type $A(1)$ and $|n_i\rangle=(001)^T$ otherwise. By now it is well established that the master equation governing the dynamics

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of the systems can be rewritten as an imaginary-time Schrödinger equation [20]:

$$\frac{\partial}{\partial t}|P(t)\rangle = -H|P(t)\rangle, \tag{1}$$

where $H$ denotes the Markov generator, also called *stochastic Hamiltonian*, and is in general neither Hermitian nor normal. Its explicit form is given below. We also introduce the *left vacuum* $\langle\widetilde{\chi}|$ which is defined by

$$\langle\widetilde{\chi}| \equiv \sum_{\{n\}} \langle\{n\}|. \tag{2}$$

Probability conservation yields the local equation (stochasticity of $H$): $\langle\widetilde{\chi}|H = \Sigma_{e^\alpha}\Sigma_{\mathbf{m}}\langle\widetilde{\chi}|H_{\mathbf{m},\mathbf{m}+e^\alpha} = 0 \Rightarrow \langle\widetilde{\chi}|H_{\mathbf{m},\mathbf{m}+e^\alpha} = 0$, where $e^\alpha$ denotes the unit vector in the direction $\alpha$ $(1 \leqslant \alpha \leqslant d)$ and $\mathbf{m}$ designates a point of the hyperlattice labeled with help of its $d$ components: $\mathbf{m} = (m_1, \ldots, m_d)$.

In this work, we assume that there are only symmetric nearest-neighbor jump processes. A particle $A$ (respectively, $B$) can jump, with rate $\Gamma_{01}^{10} = \Gamma_{10}^{01} > 0$ (respectively, $\Gamma_{02}^{20} = \Gamma_{20}^{02} > 0$) to an adjacent site (in the $d$ directions) if the latter was previously empty. Such processes are symbolized by the "reaction" $A\emptyset \leftrightarrow \emptyset A$ (respectively, $B\emptyset \leftrightarrow \emptyset B$). In addition we assume that when two different particles $A$ and $B$ are adjacent, they can cross each other with rate $\Gamma_{12}^{21} = \Gamma_{21}^{12} > 0$. These processes are schematized by the reaction $AB \leftrightarrow BA$.

The local Markov generator corresponding to this system, which acts on two adjacent sites $\mathbf{m}$ and $\mathbf{m}+e^\alpha$ reads

$$-H_{\mathbf{m},\mathbf{m}+e^\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Gamma_{01}^{01} & 0 & \Gamma_{10}^{01} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Gamma_{02}^{02} & 0 & 0 & 0 & \Gamma_{20}^{02} & 0 & 0 \\ 0 & \Gamma_{01}^{10} & 0 & \Gamma_{10}^{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Gamma_{12}^{12} & 0 & \Gamma_{21}^{12} & 0 \\ 0 & 0 & \Gamma_{02}^{20} & 0 & 0 & 0 & \Gamma_{20}^{20} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Gamma_{12}^{21} & 0 & \Gamma_{21}^{21} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

where the same notations as in Refs. [21,17] have been used. Probability conservation implies that each column in the above representation sums up to zero. Locally, the left vacuum $\langle\widetilde{\chi}|$ has the representation $\langle\widetilde{\chi}| = (111)\otimes(111)$.

The action of any operator on the left vacuum has a simple summation interpretation. This observation will be crucial in the following. Below we shall assume an initial state $|P(0)\rangle$ and investigate the expectation value of an operator $O$ (observables such as density etc.) $\langle O\rangle(t) \equiv \langle\widetilde{\chi}|Oe^{-Ht}|P(0)\rangle$.

From Eq. (3), we can compute the equations of motion of the density and of the two-point correlation functions [17]. For the density, we have

$$\frac{d}{dt}\langle n_{\mathbf{m}}^{A,B}(t)\rangle \equiv \frac{d}{dt}\langle\widetilde{\chi}|n_{\mathbf{m}}^{A,B}e^{-Ht}|P(0)\rangle = -\sum_{e^\alpha} \langle n_{\mathbf{m}}^{A,B}(H_{\mathbf{m},\mathbf{m}+e^\alpha} + H_{\mathbf{m}-e^\alpha,\mathbf{m}})\rangle(t). \tag{4}$$

For the derivation of the equation of motion of two-point correlation functions we would proceed similarly, as in [17]. One should, however, pay attention to distinguish the case of the correlation function of adjacent sites from the general case.

In general, when $\Gamma_{10}^{01}$, $\Gamma_{01}^{10}$, $\Gamma_{20}^{02}$, $\Gamma_{02}^{20}$, $\Gamma_{12}^{21}$, $\Gamma_{21}^{12}$ are independent parameters, the equations of motion of the multipoint correlation functions constitute an open hierarchy and the *dynamics* are not soluble. The stationary states of such systems have been studied in [12] by Arndt *et al.* Recently it has been shown [13] with the help of *quadratic algebra* techniques [13,12] and numerical means that an asymmetric version of this model exhibits a first-order phase transition, in its stationary state, when $\Gamma_{10}^{01} = \Gamma_{01}^{10} = \Gamma_{20}^{02} = \Gamma_{02}^{20} = \Gamma_{12}^{21} = 1$ and $\Gamma_{12}^{21} = q$. The steady state of the density of the same model has also been studied independently by Rajewsky *et al.* [14] who obtained different results; in [14], the authors, argued that there is no phase transition from the "mixed phase" to a "disordered phase."

Here we assume

$$\Gamma_{10}^{01} = \Gamma_{01}^{10} = \Gamma_{20}^{02} = \Gamma_{02}^{20} = \Gamma_{12}^{21} = \Gamma_{21}^{12} = \Gamma, \tag{5}$$

which guarantees that the equations of motion of the correlation functions close in arbitrary dimensions.

From now on we focus on the soluble model described by Eq. (3) with equal rates, according to the *solubility constraints* (5). Before studying statistical and dynamical properties of this model, let us comment on its solvability. In the single-species reaction-diffusion models, the solvability inherent to the closure of the hierarchy was explained in the framework of the duality transformations. In fact, it has been shown that the spectrum of the single-species stochastic Hamiltonian (with the solubility constraints) is identical to the spectrum of an anisotropic spin-1/2 Heisenberg quantum Hamiltonian $XXZ$ in a magnetic field [20]. As shown in [21], the situation is quite different for the multispecies problem and a general, comprehensive and unified understanding of the formal solubility is still lacking. However, for the model under consideration here, it has been shown [22] that the stochastic Hamiltonian (3), can be mapped, via a similarity transformation, to an exactly integrable quantum spin-1 model introduced by Sutherland [23].

## III. EXACT STUDY OF THE DENSITY

In this section, we study the density of the system, in particular, when translation invariance is broken (the initial density is nonuniform) and when correlations in the initial state are present.

It follows from Eq. (4) that the density of species $j \in (A,B)$ at site $\mathbf{m}$, labeled with its $d$ components $[\mathbf{m}=(m_1, \ldots, m_d)]$, obeys to the following linear differential-difference equation [17]:

$$\frac{d}{dt}\langle n_{\mathbf{m}}^{j}\rangle(t) = -2\Gamma d\langle n_{\mathbf{m}}^{j}\rangle(t) + \Gamma\sum_{\alpha=1}^{d}(\langle n_{\mathbf{m}+e^{\alpha}}^{j}\rangle + \langle n_{\mathbf{m}-e^{\alpha}}^{j}\rangle). \tag{6}$$

We first consider the situation where particles are initially nonuniformally distributed. Namely, we assume that particles of type $B$ are located in the region of space $L/2<x_1 \leqslant L, \ldots, L/2<x_d \leqslant L$ while particles of type $A$ are initially confined in the region $0 \leqslant x_1 \leqslant L/2, \ldots, 0 \leqslant x_d \leqslant L/2$ (we assume that $L$ is even). Within each of the two regions, particles of each type are distributed uniformally with respective densities $\rho_B(0)$ and $\rho_A(0)$. Solving Eq. (6) for this initial condition, we find

$$\begin{aligned}\langle n_{\mathbf{m}}^{A}(t)\rangle &= \rho_A(0)e^{-2d\Gamma t}\sum_{\mathbf{m}'}\langle n_{\mathbf{m}'}^{A}(0)\rangle\prod_{\alpha=1}^{d} I_{m_\alpha - m'_\alpha}(2\Gamma t)\\ &= \rho_A(0)e^{-2d\Gamma t}\sum_{\mathbf{m}'}\prod_{\alpha=1}^{d}\left[\Theta\left(\frac{L}{2}-m_\alpha\right)I_{m_\alpha - m'_\alpha}(2\Gamma t)\right]\\ &= \rho_A(0)\prod_{\alpha=1}^{d}\left(\sum_{0\leqslant m'_\alpha \leqslant L/2} e^{-2\Gamma t}I_{m_\alpha - m'_\alpha}(2\Gamma t)\right),\end{aligned} \tag{7}$$

where $I_n(z)$ denotes the usual modified Bessel function. We have also introduced the Heaviside function $\Theta(x)=0$ if $x<0$ and $\Theta(x)=1$ if $x>1$.

With help of the asymptotic expansion of the Bessel functions and approximating $\Sigma_{n_\alpha} e^{-n_\alpha^2/4\Gamma t} \approx \int dn_\alpha e^{-n_\alpha^2/4\Gamma t}$, we obtain from Eq. (8) the long-time behavior of the density at site $\mathbf{m}$:

$$\langle n_{\mathbf{m}}^{A}(t)\rangle \approx \rho_A(0)\prod_{\alpha=1}^{d}\left[\frac{\mathrm{erf}\left(\frac{L/2-m_\alpha}{\sqrt{4\Gamma t}}\right)-\mathrm{erf}\left(\frac{m_\alpha}{\sqrt{4\Gamma t}}\right)}{2}\right], \tag{8}$$

where $\mathrm{erf}(z)$ denotes the usual error function.

Similarly, we have for the density of particles $B$:

$$\begin{aligned}\langle n_{\mathbf{m}}^{B}(t)\rangle &= \rho_B(0)\prod_{\alpha=1}^{d}\left(\sum_{L/2<m'_\alpha \leqslant L} e^{-2\Gamma t}I_{m_\alpha - m'_\alpha}(2\Gamma t)\right)\\ &\approx \rho_B(0)\prod_{\alpha=1}^{d}\left[\frac{\mathrm{erf}\left(\frac{L-m_\alpha}{\sqrt{4\Gamma t}}\right)-\mathrm{erf}\left(\frac{L/2-m_\alpha}{\sqrt{4\Gamma t}}\right)}{2}\right].\end{aligned} \tag{9}$$

We now pass to the case where the distribution of particles for each species $j \in (A,B)$ is given by ($\kappa_j$ denotes a real dimensionless constant) an initially correlated distribution:

$$\langle n_{\mathbf{m}}^{j}\rangle(0) = \rho_j(0)\left(\prod_{i=1\ldots d}\delta_{m_i,0} + \kappa_j \times \prod_{i=1\ldots d}|m_i|^{-\gamma_i}(1-\delta_{m_i,0})\right). \tag{10}$$

The exact densities then read [17]

$$\langle n_{\mathbf{m}}^{j}\rangle(t) = \rho_j(0)\left[\prod_{i=1\ldots d}(e^{-4\Gamma t}I_{m_i}(2\Gamma t)) + \kappa_j\prod_{i=1\ldots d}\left(e^{-2\Gamma t}\sum_{m'_i\neq 0}|m'_i|^{-\gamma_i}I_{m_i-m'_i}(2\Gamma t)\right)\right]. \tag{11}$$

For $\kappa_j \neq 0$, in the limit $m \sim L \gg 1$ and $\Gamma t \gg 1$, with $\sigma \equiv m/L$ and $u \equiv L^2/4\Gamma t$. When $\sigma = \mathcal{O}(1)$, then $u \sim \sigma^2 L^2/4\Gamma t = m^2/4\Gamma t$, and we obtain

$$\langle n_{\mathbf{m}}^{j}\rangle(t) \sim \begin{cases} \displaystyle\prod_{i=1}^{d}\frac{2e^{-\sigma^2 u}}{(1-\gamma_i)(4u\sigma^2\Gamma t)^{\gamma_i/2}} & \text{if} \quad 0 \leqslant \gamma_i < 1,\\ \displaystyle\prod_{i=1}^{d}\frac{(1+e^{-\sigma^2 u})\zeta(\gamma_i)}{(\pi\Gamma t)^{1/2}} & \text{if} \quad \gamma_i > 1,\\ \displaystyle\frac{(e^{-\sigma^2 u}\ln(4u\sigma\Gamma t))^d}{(\pi\Gamma t)^{d/2}} & \text{if} \quad \gamma_i = 1,\end{cases} \tag{12}$$

where $\zeta(\nu)=\Sigma_{k\geqslant 1}k^{-\nu},\nu>1$ is the Riemann zeta function.

It follows from these results that for $\gamma_i\neq 1$, the density decays as a power law of time. However, notice that when initial correlations are *strong* (i.e., $0<\gamma_i<1$), the algebraic decay of the density is *nonuniversal* (it depends of $\gamma_i$). When initial correlations are *weak* (i.e., $\gamma_i>1$), the algebraic decay of the density is *universal*. Hence the case where $\gamma_i=1$ is *marginal* and there are logarithmic corrections to the *universal behavior*.

For initial states decribed by Eq. (10), with $\gamma_i=0$ and $\kappa_j\neq 0$, then we have

$$\langle n_{\mathbf{m}}^j\rangle(t)\sim\rho_j(0)\left[\frac{e^{-d\sigma^2 u}}{(4\pi\Gamma t)^{d/2}}+\left\{\frac{\Gamma(1/2)-\Gamma(1/2,u)}{\sqrt{4\pi}}\right\}k_j\right.\\ \left.\times\left(1+\frac{1}{8\Gamma t}\right)\right]. \tag{13}$$

Therefore the dimensionality has a nontrivial effect, when $d<2$, the densities decay as $t^{-d/2}$. Otherwise, when $d>2$, $\langle n_{\mathbf{m}}^j\rangle(t)\sim t^{-1}$.

On the other hand, when $\kappa_j=0$, the initial density of species $j$ vanishes on the hypercube except at the origin, where its value is $\rho_j(0)$. In this case, the limit considered above, yields

$$\langle n_{\mathbf{m}}^j\rangle(0)\sim\frac{e^{-d\sigma^2 u}}{(4\pi\Gamma t)^{d/2}}. \tag{14}$$

Notice that because of conservation of the number of particles, in the *translationally invariant situation*, we simply have

$$\rho_A(t)=\rho_A(t=0)\equiv\rho_A,\quad \rho_B(t)=\rho_B(t=0)\equiv\rho_B. \tag{15}$$

## IV. INSTANTANEOUS TWO-POINT CORRELATION FUNCTIONS FOR TRANSLATIONALLY INVARIANT SYSTEMS

In this section we compute exactly the two-point correlation function for translationally invariant systems, in arbitrary dimensions for different initial states.

The equations of motion for the connected correlation functions $\mathcal{C}_{\mathbf{r}}^{ij}(t)\equiv\mathcal{C}_{-\mathbf{r}}^{ij}(t)\equiv\langle n_{\mathbf{l}}^i n_{\mathbf{m}}^j\rangle(t)-\rho_i\rho_j=\langle n_0^i n_{\mathbf{m-l}}^j\rangle(t)-\rho_i\rho_j,(i,j)\in(A,B)$, read, with the notation: $\mathbf{r}=(r_1,\ldots,r_\alpha,\ldots r_d)\equiv\mathbf{m}-\mathbf{l}$, where $\alpha=1,\ldots,d$.

$$\frac{\partial}{\partial t}\mathcal{C}_{\mathbf{r}}^{ij}(t)=-4\Gamma d\mathcal{C}_{\mathbf{r}}^{ij}(t)+2\Gamma\\ \times\sum_{\alpha=1}^{d}\left(\mathcal{C}_{\mathbf{r}+e^{\alpha}}^{ij}(t)+\mathcal{C}_{\mathbf{r}-e^{\alpha}}^{ij}(t)\right),\quad ||\mathbf{r}||\geqslant 2,$$

$$\frac{\partial}{\partial t}\mathcal{C}_{e^{\alpha}}^{ij}(t)=2\Gamma\left[\mathcal{C}_{2e^{\alpha}}^{ij}(t)+\sum_{\alpha'\neq\alpha=1\ldots d}\{\mathcal{C}_{e^{\alpha}-e^{\alpha'}}^{ij}(t)\right.\\ \left.+\mathcal{C}_{e^{\alpha}+e^{\alpha'}}^{ij}(t)\}-(2d-1)\mathcal{C}_{e^{\alpha}}^{ij}(t)\right],$$

$$\frac{\partial}{\partial t}\mathcal{C}_0^{ij}(t)=0. \tag{16}$$

Solving the latter, we have, using known properties of modified Bessel functions [see the Appendix, Eq. (A1)]:

$$\mathcal{C}_{\mathbf{r}}^{ij}(t)=\sum_{\mathbf{r}'\neq 0}\mathcal{C}_{\mathbf{r}'}^{ij}(0)e^{-4d\Gamma t}\prod_{\alpha=1}^{d}I_{r_\alpha-r'_\alpha}(4\Gamma t)\\ +\mathcal{C}_0^{ij}(0)e^{-4d\Gamma t}\prod_{\alpha=1}^{d}I_{r_\alpha}(4\Gamma t)-\int_0^t d\tau e^{-4d\Gamma\tau}\\ \times\mathcal{C}_{e^{\alpha}}^{ij}(t-\tau)\left(\frac{\partial}{\partial\tau}-4d\Gamma\right)\prod_{\alpha=1}^{d}I_{r_\alpha}(4\Gamma\tau)\\ -4d\Gamma\int_0^t d\tau e^{-4d\Gamma\tau}\mathcal{C}_{e^{\alpha}}^{ij}(t-\tau)\prod_{\alpha=1}^{d}I_{r_\alpha}(4\Gamma\tau)\\ +2\Gamma\int_0^t d\tau e^{-4d\Gamma\tau}\sum_{\alpha=1}^{d}\mathcal{C}_{e^{\alpha}}(t-\tau)[I_{r_\alpha+1}(4\Gamma\tau)\\ +I_{r_\alpha-1}(4\Gamma\tau)]\prod_{\alpha'\neq\alpha}I_{r_{\alpha'}}(4\Gamma\tau). \tag{17}$$

Restricting the solution to *one-spatial dimension*, with $r\equiv|m-l|>0$, the Laplace transform yields $[i,j\in(A,B)]$

$$\mathcal{C}_1^{ij}(t)=e^{-4\Gamma t}\sum_{r'\geqslant 1}\mathcal{C}_{r'}^{ij}(0)\{I_{r'}(4\Gamma t)+I_{r'-1}(4\Gamma t)\} \tag{18}$$

and more generally,

$$\mathcal{C}_{r\geqslant 1}^{ij}(t)=e^{-4\Gamma t}\sum_{r'\geqslant 1}\mathcal{C}_{r'}^{ij}(0)\{I_{r'+r-1}(4\Gamma t)+I_{r'-r}(4\Gamma t)\}. \tag{19}$$

Let us now consider *one-spatial dimension* and assume that the initial correlations are given by

$$\mathcal{C}_r^{l}(0)=\kappa_l r^{-\nu_l},\quad \nu_l\geqslant 0,\quad l\in(AA,BB,AB). \tag{20}$$

We discuss the case $|\kappa_l|>0$ while the case $\kappa_l=0$ corresponds either to the situation where no particle is present on the lattice initially, or, when all sites of the lattice are occupied by particles of species $i$ (or $j$). An alternative is that the system would be initially in its steady state. When a single species is present initially, say species $A$, we recover the known problem of symmetric diffusion of hard particles $A+\varnothing\leftrightarrow\varnothing+A$. When the lattice is full (or empty) initially, no dynamics takes place. A single-species one-dimensional process $A+A\leftrightarrow\emptyset+\emptyset$ with a correlated initial state as in Eq. (20) has been studied in [18].

Again we can infer the asymptotic behavior of the two-point connected correlation functions in the limit $\Gamma t\gg 1$ with $v\equiv L^2/8\Gamma t<\infty$ [17].

It is useful for the sequel to introduce the definitions of the auxiliary functions [17]:

$$\mathcal{F}_1(v,\sigma,\nu_l)\equiv\frac{\left\{\Gamma\left(\frac{1-\nu_l}{2}\right)+\Gamma\left(\frac{1-\nu_l}{2},\sigma^2 v\right)-\Gamma\left[\frac{1-\nu_l}{2},v(1-\sigma)^2\right]-\Gamma\left[\frac{1-\nu_l}{2},v(1+\sigma)^2\right]\right\}}{\sqrt{4\pi}}, \tag{21}$$

$$\mathcal{F}_2(v,\sigma,\nu_l)\equiv\sqrt{\frac{v}{\pi}}\left\{\frac{e^{-\sigma^2 v}}{1-\nu_l}+1\right\}. \tag{22}$$

We distinguish two regimes

(i) For $r\ll L$, with $r_\alpha^2/8\Gamma t\ll 1$ and $\sigma\equiv r/L$,

$$\mathcal{C}_r^l(t)\sim\begin{cases}\dfrac{\kappa_l\mathcal{F}_1(v,\sigma,\nu_l)}{\sqrt{4\pi}(8\Gamma t)^{\nu_l/2}} & \text{if}\quad 0\leqslant\nu_l<1,\\[2ex] \dfrac{\kappa_l[2\zeta(\nu_l)]}{(8\pi\Gamma t)^{1/2}} & \text{if}\quad \nu_l>1,\\[2ex] \dfrac{\kappa_l\ln[8\Gamma v(1-\sigma_\alpha)t]}{(8\pi\Gamma t)^{1/2}} & \text{if}\quad \nu_l=1.\end{cases} \tag{23}$$

(ii) For $r\gg 1, r\equiv\sigma L\sim L$, we have

$$\mathcal{C}_r^l(t)\sim\begin{cases}\kappa_l\left(\dfrac{\mathcal{F}_2(v,\sigma,\nu_l)}{\sqrt{4\pi}(8\Gamma t)^{\nu_l/2}}\right) & \text{if}\quad 0\leqslant\nu_l<1,\\[2ex] \dfrac{\kappa_l[(1+e^{-\sigma^2 v})\zeta(\nu_l)]}{(8\pi\Gamma t)^{1/2}} & \text{if}\quad \nu_l>1,\\[2ex] \dfrac{\kappa_l e^{-\sigma^2 v}\ln(8\Gamma v\sigma t)}{(8\pi\Gamma t)^{1/2}} & \text{if}\quad \nu_l=1.\end{cases} \tag{24}$$

As for the density, it follows from these results that for $\nu_l\neq 1$, the (connected-)correlation functions decay as a power law of time. When initial correlations are *strong* (i.e., $0<\nu_l<1$) the power-law decay of the correlation functions is *nonuniversal* . In contrast, when initial correlations are *weak* (i.e., $\nu_l>1$), the algebraic decay of the correlation functions is *universal*. The case where $\nu_l=1$ is *marginal* and logarithmic corrections to the *universal behavior* arise.

In arbitrary dimension ($d\geqslant 1$), we consider a translationally invariant random but uncorrelated intial state, described by

$$\mathcal{C}_{\mathbf{r}}^l(0)=\kappa_l, l\in(AA,BB,AB), \tag{25}$$

where, as above, $\kappa_l\neq 0$.

The asymptotic behavior ($L,\Gamma t\gg 1$ with $v\equiv L^2/8\Gamma t<\infty$ and $\sigma_\alpha\equiv r_\alpha/L$) of the connected correlation functions is then [17]:

$$\mathcal{C}_{\mathbf{r}}^l(t)\sim\begin{cases}\kappa_l\left(1+\dfrac{1}{16\Gamma t}\right)^d\displaystyle\prod_{\alpha=1}^{d}\mathcal{F}_{1,\alpha}(v,\sigma_\alpha,\nu_l=0), & r_\alpha\ll L,\\[2ex] \kappa_l\left(1+\dfrac{1}{16\Gamma t}\right)^d\displaystyle\prod_{\alpha=1}^{d}\mathcal{F}_{2,\alpha}(v,\sigma_\alpha,\nu_l=0), & r_\alpha\sim L\gg 1,\end{cases} \tag{26}$$

where the quantities $\mathcal{F}_{1,\alpha}$ and $\mathcal{F}_{2,\alpha}$ are obtained, respectively, from Eqs. (21) and (22) on the substitution of $\sigma$ by $\sigma_\alpha \equiv r_\alpha/L$.

## V. THE MAPPING ONTO A RESTRICTED SOLID-ON-SOLID GROWTH MODEL WITH "THREE STATES"

In this section, we introduce a growth model of *restricted solid-on-solid* (RSOS) type with "three states," by exploiting a mapping of the one-dimensional model studied in the previous sections.

Let us briefly recall that the RSOS growth models are, e.g., useful to describe the spatial fluctuations of the (one-dimensional, of length $L$) interface location in the magnetization profile between coexistent phases in two-dimensional models of ferromagnets, such as in the zero-field planar Ising model [2]. In such models, at zero temperature, every path minimizing the energy of the systen is a sequence of $L$ binary numbers $n_j=0,1$ with $j=1,\dots,L$. The stochastic variable $n_j$ has the value $n_j=0$ if the $j$th segment of the interface steps upwards (in an angle of $\pi/4$). The value $n_j=1$ corresponds to the case where the segments steps downward with an angle of $\pi/4$. The quantities $n_j=0,1$ can be interpreted as occupation numbers relating the interface height $h_j$ according to $h_j-h_{j-1}=1-2n_j$ [2]. In this case the displacement $\delta h_r(t)$, at time $t$ of the segment of the interface from the sites $j_1$ to $j_2>j_1$, with $r\equiv j_2-j_1>0$ is given by $\delta h_r(t) = \Sigma_{k=1}^r[1-2n_k(t)]$ and thus $|\delta h_{r+1}-\delta h_r|=1$.

Here we consider an extension of the above model. We consider that the configurations minimizing the energy (at zero temperature) are of the form $\{x_A n_1^A + x_B n_1^B, x_A n_2^A + x_B n_2^B, \dots, x_A n_L^A + x_B n_L^B\}$, where the discrete stochastic variable $x_A n_j^A + x_B n_j^B$ can take *three* values. The case $x_A n_j^A + x_B n_j^B=0$ again corresponds to the situation where the $j$th segment steps upwards with an angle of $\pi/4$. The case $x_A n_j^A + x_B n_j^B = x_A$ (respectively, $x_A n_j^A + x_B n_j^B = x_B$) describes the situation where the $j$th segment forms an angle $\arctan(1-x_A)$ [respectively, $\arctan(1-x_B)$] with the horizontal. When $x_B=0$ (respectively, $x_A=0$) and $x_A=2$ (respectively, $x_B=2$), we recover the above-mentioned two-state RSOS growth model. In this sense the growth model which we study hereafter is a "three-states" extension of the usual RSOS model [2,7,18,19]. In addition the mapping with the diffusive model is clear, the presence of particle of species $A$ (respectively, $B$) at site $j$ translates in the language of the growth model with the fact that the related segment of the interface forms an angle $\arctan(1-x_A)$ (respectively, $\arctan(1-x_B)$) with the horizontal. In this picture, the jumping of the diffusive particles correponds to the fluctuation of the orientation of the related segments of the interface.

We consider a translationally invariant system; the displacement of the (one-dimensional) interface, at time $t$, from the sites $k_1$ to $k_2>k_1$, with $r=k_2-k_1>0$ is given by $h_{k_2}(t)-h_{k_1}(t)\equiv\delta h_r(t)$, where

$$\delta h_r(t) \equiv \sum_{m=1}^{r} [1-x_A n_m^A(t) - x_B n_m^B(t)]. \tag{27}$$

Therefore, in the model considered here, we have $\max|\delta h_{r+1}(t) - \delta h_r(t)| = \max(|x_A-1|,|x_B-1|,1)$, instead of the usual constraint $|\delta h_{r+1}(t)-\delta h_r(t)|=1$ of the conventional RSOS models. The mean displacement of the interface reads $\langle \delta h_r(t)\rangle = r(1-x_A\rho_A - x_B\rho_B)$ thus, if one wants to impose a zero mean displacement of the interface , we have to require that $x_A\rho_A + x_B\rho_B=1$.

In this section we are interested in the computation of the fluctuations of $\delta h_j(t)$:

$$\begin{aligned}\langle[\delta h_r(t)]^2\rangle \equiv w^2(r,t) = {} & [(x_A+x_B)^2-1]r^2 \\ & + \sum_{r'=1}^{r}\sum_{l=1}^{r-r'+1}[x_A^2\langle n_l^A n_{l+r'}^A\rangle(t) \\ & + x_B^2\langle n_l^B n_{l+r'}^B\rangle(t) + 2x_A x_B\langle n_l^A n_{l+r'}^B\rangle(t)],\end{aligned} \tag{28}$$

where $w(r,t)$ is the physical width of the interface.

Using the fact that $[(i,j)\in(A,B)]$ $\partial/\partial t\Sigma_{l=1}^r\langle n_m^i n_{m+l}^j\rangle \times(t) = 2\Gamma[\langle n_m^i n_{m+r+1}^j\rangle(t) - \langle n_m^i n_{m+r}^j\rangle(t)]$ and $\Sigma_{s=1}^r \times \Sigma_{l=1}^{r-s}\partial/\partial t\langle n_m^i n_{m+l}^j\rangle(t) = 2\Gamma[\langle n_m^i n_{m+r+1}^j\rangle(t) - \langle n_m^i n_{m+1}^j\rangle \times(t)]$, we obtain the following equation of motion for the width:

$$\begin{aligned}\frac{\partial w^2(r,t)}{\partial t} = {} & 4\Gamma[x_A^2\{\mathcal{C}_r^{AA}(t)-\mathcal{C}_1^{AA}(t)\} + x_B^2\{\mathcal{C}_r^{BB}(t)-\mathcal{C}_1^{BB}(t)\} \\ & + 2x_A x_B\{\mathcal{C}_r^{AB}(t)-\mathcal{C}_1^{AB}(t)\}], \quad r>1,\end{aligned} \tag{29}$$

with $w^2(1,t)=w^2(1,t=0)=(x_A+x_B)^2+1-2(\rho_A x_A + \rho_B x_B)$.

For $r>1$, we thus have

$$\begin{aligned}w^2(r,t) = {} & w^2(r,0) + 4\Gamma\int_0^t dt'[x_A^2\{\mathcal{C}_r^{AA}(t')-\mathcal{C}_1^{AA}(t')\} \\ & + x_B^2\{\mathcal{C}_r^{BB}(t')-\mathcal{C}_1^{BB}(t')\} + 2x_A x_B\{\mathcal{C}_r^{AB}(t') \\ & - \mathcal{C}_1^{AB}(t')\}],\end{aligned} \tag{30}$$

where

$$\begin{aligned}w^2(r,0) = {} & [(x_A+x_B)^2+1-2(\rho_A x_A+\rho_B x_B)]r^2 \\ & + 2\sum_{r'=1}^{r}(r-r')[x_A^2\langle n_m^A n_{m+r'}^A\rangle(0) \\ & + x_B^2\langle n_m^B n_{m+r'}^B\rangle(0) + 2x_A x_B\langle n_m^A n_{m+r'}^B\rangle(0)].\end{aligned} \tag{31}$$

With help of the formula [24]: $4\Gamma\int_0^t dt' e^{-4\Gamma t'} I_n(4\Gamma t') = 4\Gamma t e^{-4\Gamma t}[I_0(4\Gamma t)+I_1(4\Gamma t)] + n[e^{-4\Gamma t}I_0(4\Gamma t)-1] + 2e^{-4\Gamma t}\sum_{k=1}^{n-1}(n-k)I_k(4\Gamma t)$, and with the explicit expression of the correlation functions (18) and (19), we obtain the following exact expression for $w(r,t)$:

$$w^2(r,t)-w^2(r,0) = 2e^{-4\Gamma t}\sum_{r'\geqslant 1}[x_A^2\mathcal{C}_{r'}^{AA}(0)+x_B^2\mathcal{C}_{r'}^{BB}(0)+2x_Ax_B\mathcal{C}_{r'}^{AB}(0)] \times\left[\sum_{k=1}^{r+r'-2}(r+r'-k-1)I_k(4\Gamma t) + \sum_{k=1}^{r+r'-1}(r'-k-r)I_k(4\Gamma t) - 2\sum_{k=1}^{r'-2}(r'-k)I_k(4\Gamma t) - I_{r'-1}(4\Gamma t)\right]. \tag{32}$$

We will now specifically focus on two kinds of initial states:

(i) We assume first that the system is initially characterized by an alternating periodic array of particles of type *A* and *B*. We consider thus an initial state $|P(0)\rangle = 1/2(|ABAB\ldots\rangle + |BABA\ldots\rangle)$, with $x_A+x_B=2$ (in this case $\langle\delta h_r(t)\rangle=0$). This initial *flat interface* leads to $\rho_A=\rho_B=1/2$, and for the connected initial correlation functions, we have $\mathcal{C}_r^{AA}(0)=(-1)^r/4=\mathcal{C}_r^{BB}(0)=-\mathcal{C}_r^{AB}(0)$.

It follows from Eq. (31) that the initial fluctuations in this case read $w^2(r,0)=r[4r+\frac{3}{4}(x_A-x_B)^2-1]$, for $r>1$ and $r$ even. Therefore, we also have $w^2(1,t)=3$.

For this initial configuration, the expression (18) simplifies and we have $\mathcal{C}_1^{AA}(t)=\mathcal{C}_1^{BB}(t)=-\mathcal{C}_1^{AB}(t)=-(e^{-4\Gamma t}/4)I_0(4\Gamma t)$. The general expression (32) of the fluctations reads $w^2(r,t)-w^2(r,0)=(x_A-x_B)^2\Gamma\int_0^t dt' e^{-4\Gamma t'}\{I_0(4\Gamma t')+\sum_{r'\geqslant 1}(-1)^{r'}[I_{r+r'-1}(4\Gamma t')+I_{r-r'}(4\Gamma t')]\}$. From this expression, using the asymptotic behavior of the Bessel functions, it is possible to obtain the long-time behavior (for $\Gamma t\gg 1$, $r\gg 1$) of the fluctuations

$$w(r,t)^2-w(r,0)^2 = (x_A-x_B)^2\sqrt{\frac{\Gamma t}{2\pi}}\{1+\mathcal{O}[(\Gamma t)^{-1}]\}+(x_A-x_B)^2 \times\sum_{r'\geqslant 1}(-1)^{r'}\int_0^t\frac{dt'}{\sqrt{8\pi\Gamma t'}} \times\{\exp[-(r+r'-1)^2/8\pi\Gamma t'] + \exp[-(r-r')^2/8\pi\Gamma t']\}\{1+\mathcal{O}[(r-r')^{-2}]\}. \tag{33}$$

Thus for the initial condition considered here, when $x_A\neq x_B$, it follows from Eq. (33) that the fluctuations grow as $w(r,t)\approx(\Gamma t)^{1/4}$.

On the other hand, it is known that for the *flat interface* (or "sawtooth" initial state), whose dynamics are coded in a two-state model $\delta h_r=\sum_{m=0}^n(1-2n_m^A)$, the fluctuations grow as $\sim(\Gamma t)^{1/4}$ [7,18,2]. In the situation considered here, the fluctuations still grow as $w(r,t)\sim(\Gamma t)^{1/4}$ and thus the details of the model and its "three-state" character only appears through the amplitude $(x_A-x_B)^2$. When $x_A=x_B=1$, the initial configuration corresponds to a straight line and, according to Eq. (33), there are no fluctuations. Let us also note that when, e.g., $x_A=2$ and $x_B=0$, the *B* particles play the role of the vacancies in the two-state model and the model (27) is exactly mapped onto the well-studied two-state model [7,18,2].

(ii) We now investigate the fluctuations of $\delta h_r(t)$ in the presence of initial correlations.

We assume that particles of type *A* and *B* are distributed according to [18]

$$\mathcal{C}_r^l(0)=\kappa_l r^{-\nu_l}, \nu_l\geqslant 0, l\in(AA,BB,AB), \tag{34}$$

which corresponds, via the mapping (27), in the language of the growth model, to an interface with initial fluctuations given by Eq. (31), where $\langle n_m^i n_{m+r'}^j\rangle(0)=\mathcal{C}_{r'}^{ij}(0)+\rho_i\rho_j$, $(i,j)\in(A,B)$.

Let us define the following quantity:

$$\nu=\min(\nu_{AA},\nu_{BB},\nu_{AB}). \tag{35}$$

With help of Eq. (29) and Eqs. (20) and (23), we can compute the asymptotic expression of the fluctuations for $\Gamma t\gg 1$ and $r\gg 1$, with $\upsilon\equiv L^2/8\Gamma t$ and $\sigma\equiv r/L=\mathcal{O}(1)$ (we assume that $x_Ax_B\neq 0$), which reads

$$w(r,t)^2-w(r,0)^2\sim\begin{cases}\dfrac{2\Gamma t}{\sqrt{\pi}\nu(8\pi\Gamma t)^{\nu/2}}[\mathcal{F}_1(\upsilon,\sigma,\nu)-\mathcal{F}_2(\upsilon,\sigma,\nu)] & \text{if} \quad 0<\nu<1,\\ \sqrt{2\Gamma t}\,\xi(\nu)(1-e^{-\sigma^2\upsilon}) & \text{if} \quad \nu>1,\\ \sqrt{\dfrac{2\Gamma t}{\pi}}[\ln(8\Gamma\upsilon t)-e^{-\sigma^2\upsilon}\ln(8\Gamma\upsilon\sigma t)] & \text{if} \quad \nu=1,\end{cases} \tag{36}$$

where the quantities $\mathcal{F}_1$ and $\mathcal{F}_2$ have been defined in Eqs. (21), (22), and $\xi(\nu)$ is the usual Riemann zeta function.

We see that in the presence of initial correlations (34), the fluctuations are dominated by the smallest initial correlation exponent $\nu$. Therefore, if $\nu=\nu_{AA}$ (respectively, $\nu=\nu_{BB}$), the dominant contribution (36) to the fluctuations are the same as for a correlated two-state model where $x_A=2$ and $x_B=0$ (respectively, $x_A=0$ and $x_B=2$) where the $B$ (respectively, the $A$) particles play the role of vacancies [18].

From Eq. (36) we see that initial correlations affect the long-time behavior of the fluctuations of the height displacement of the interface when the correlations are "strong enough" (i.e., $0<\nu<1$), thus $w(r,t)\sim(\Gamma t)^{1/2-\nu/4}$. Conversely, for "weak" initial correlations ($\nu>1$) we recover the usual fluctuation exponent: $w(r,t)\sim(\Gamma t)^{1/4}$. The intermediate case $\nu=1$, corresponds to the *marginal* behavior where $w(r,t)\sim(\Gamma t)^{1/4}\sqrt{\ln \Gamma t}$.

For the sake of completness, and to conclude this section, we compute the spatiotemporal height correlations $\langle[h_{r_1}(t)-h_{r_2}(0)]^2\rangle$, with $1\ll r_1\ll r_2\lesssim L$ for the initial state (34). To do this we have to anticipate some results of the Sec. VI. In fact one computes

$$\begin{aligned}\langle[h_{r_1}(t)-h_{r_2}(0)]^2\rangle-\langle[h_{r_1}(0)-h_{r_2}(0)]^2\rangle\\ =[w^2(r_1,t)-w^2(r_1,0)]-2\sum_{j=1}^{r_1}\sum_{k=1}^{r_2}x_A^2[\langle n_j^A(t)n_k^A(0)\rangle\\ -\langle n_j^A(0)n_k^A(0)\rangle]-2\sum_{j=1}^{r_1}\sum_{k=1}^{r_2}\{x_B^2[\langle n_j^B(t)n_k^B(0)\rangle\\ -\langle n_j^B(0)n_k^B(0)\rangle]+2x_Ax_B[\langle n_j^A(t)n_k^B(0)\rangle\\ -\langle n_j^A(0)n_k^B(0)\rangle]\},\end{aligned}\tag{37}$$

where the expressions of $\langle n_j^{A,B}(t)n_k^{A,B}(0)\rangle$ are given in Eqs. (49)–(51). The asymptotic behavior ($\Gamma t\gg1$, $1\ll r_1\ll r_2\lesssim L$) of Eq. (37) is obtained with the help of Eqs. (49)–(51) and reads [26]: $\langle[h_{r_1}(t)-h_{r_2}(0)]^2\rangle-\langle[h_{r_1}(0)-h_{r_2}(0)]^2\rangle\sim[w^2(r_1,t)-w^2(r_1,0)]$. Therefore, the long-time behavior of $\langle[h_{r_1}(t)-h_{r_2}(0)]^2\rangle$ also obeys Eq. (36), where, in this case $\sigma\equiv r_1/L$.

It is interesting at this point to compare the results obtained here and the Family-Wilcsek Ansatz (FWA), developed for the description of a ballistic deposition model, which predicts $w^2(L,t)\approx t^{\alpha/z}f(L^2/t^{2/z})$ [25,26]. Here, because of the diffusive nature of the model, $z=2$ and thus we get (i) for the flat interface an exponent $\alpha=1$, and for the case (ii) $\alpha=\max(1,2-\nu)$. In both cases (i) and (ii), when $\nu>1$, in the regime $L^2\gg\Gamma t$ (i.e., $v\gg1$) we obtain a scaling function $f(L^2/t)\approx$const, in agreement with FWA.

## VI. NONINSTANTANEOUS TWO-POINT CORRELATION FUNCTIONS

In this section, we compute exactly the noninstantaneous two-point correlation functions for various initial states. Similar quantities have already been computed, for some specific *single-species* models (see, e.g., [18,27–29]).

Let us first consider an uncorrelated initial distribution

$$|P(0)\rangle=\begin{pmatrix}1-\rho_A(0)-\rho_B(0)\\ \rho_A(0)\\ \rho_B(0)\end{pmatrix}^{\otimes L^d},$$

such that

$$\begin{aligned}\langle n_{\mathbf{m}}^A(0)n_{\mathbf{l}}^A(0)\rangle&=\rho_A(0)\delta_{\mathbf{m},\mathbf{l}}+\rho_A(0)^2(1-\delta_{\mathbf{m},\mathbf{l}}),\\ \langle n_{\mathbf{m}}^B(0)n_{\mathbf{l}}^B(0)\rangle&=\rho_B(0)\delta_{\mathbf{m},\mathbf{l}}+\rho_B(0)^2(1-\delta_{\mathbf{m},\mathbf{l}}),\\ \langle n_{\mathbf{m}}^A(0)n_{\mathbf{l}}^B(0)\rangle=\langle n_{\mathbf{m}}^B(0)n_{\mathbf{l}}^A(0)\rangle&=\rho_A(0)\rho_B(0)(1-\delta_{\mathbf{m},\mathbf{l}}).\end{aligned}\tag{38}$$

We then have

$$\begin{aligned}\langle n_{\mathbf{m}}^A(t)n_{\mathbf{l}}^A(0)\rangle=\rho_A^2(0)+(\rho_A(0)-\rho_A^2(0))\\ \times\prod_{\alpha=1\dots d}e^{-2\Gamma t}I_{m_\alpha-l_\alpha}(2\Gamma t),\end{aligned}\tag{39}$$

$$\begin{aligned}\langle n_{\mathbf{m}}^A(t)n_{\mathbf{l}}^B(0)\rangle\\ =\rho_A(0)\rho_B(0)\left[1-\prod_{\alpha=1\dots d}e^{-2\Gamma t}I_{m_\alpha-l_\alpha}(2\Gamma t)\right]\\ =\langle n_{\mathbf{m}}^B(t)n_{\mathbf{l}}^A(0)\rangle,\end{aligned}\tag{40}$$

$$\begin{aligned}\langle n_{\mathbf{m}}^B(t)n_{\mathbf{l}}^B(0)\rangle=\rho_B^2(0)+[\rho_B(0)-\rho_B^2(0)]\\ \times\prod_{\alpha=1\dots d}e^{-2\Gamma t}I_{m_\alpha-l_\alpha}(2\Gamma t).\end{aligned}\tag{41}$$

We are interested in the asymptotic behavior ($\Gamma t\gg1$ and $u=L^2/4\Gamma t<\infty$) of the above functions in the two regimes:

(i) $|m_\alpha-l_\alpha|\equiv r_\alpha\sim L\gg1$, in this case $\sigma_\alpha=r_\alpha/L=\mathcal{O}(1)$.

(ii) $|m_\alpha-l_\alpha|\equiv r_\alpha\ll L$, in this case $\sigma_\alpha=r_\alpha/L=\mathcal{O}(1/L)$.

It is worth noting that the autocorrelation functions are obtained in the second regimes (ii). We then have

$$\begin{aligned}\langle n_{\mathbf{m}}^A(t)n_{\mathbf{l}}^A(0)\rangle\\ =\rho_A^2(0)+\frac{[\rho_A(0)-\rho_A^2(0)]e^{-\sum_{\alpha=1}^{d}\sigma_\alpha^2u}}{(4\pi\Gamma t)^{d/2}}+\mathcal{O}(1/t^d),\end{aligned}\tag{42}$$

$$\begin{aligned}\langle n_{\mathbf{m}}^A(t)n_{\mathbf{l}}^B(0)\rangle=\rho_A(0)\rho_B(0)\left[1-\frac{e^{-\sum_{\alpha=1}^{d}\sigma_\alpha^2u}}{(4\pi\Gamma t)^{d/2}}+\mathcal{O}(1/t^d)\right]\\ =\langle n_{\mathbf{m}}^B(t)n_{\mathbf{l}}^A(0)\rangle,\end{aligned}\tag{43}$$

$$\langle n^B_{\mathbf{m}}(t)n^B_{\mathbf{l}}(0)\rangle = \rho_B^2(0)+\frac{(\rho_B(0)-\rho_B^2(0))e^{-\sum_{\alpha=1}^{d}\sigma_\alpha^2 u}}{(4\pi\Gamma t)^{d/2}}+\mathcal{O}(1/t^d). \tag{44}$$

In these regimes, we have a power-law decay of correlation functions $(i,j)\in(A,B))$ namely,

$$\langle n^i_{\mathbf{m}}(t)n^j_{\mathbf{l}}(0)\rangle \sim (\Gamma t)^{-d/2}e^{-\sum_{\alpha=1}^{d}\sigma_\alpha^2 u}. \tag{45}$$

Let us now pass to the case where the initial state is correlated according to

$$\langle n^i_{\mathbf{m}}(0)n^j_{\mathbf{l}}(0)\rangle = \mathcal{K}_{ij}\prod_{\alpha=1,\dots,d}(1-\delta_{r_\alpha,0})|r_\alpha|^{-\Delta^\alpha_{ij}},$$
$$r_\alpha \equiv |m_\alpha - l_\alpha|,\quad \Delta^\alpha_{ij}>0,\quad (ij)\in(A,B),$$
$$\mathcal{K}_{ij}>0,\mathrm{dist}(l,m)>0, \tag{46}$$

and

$$\langle n^i_{\mathbf{m}}(0)n^j_{\mathbf{m}}(0)\rangle = \rho_i(0)\rho_j(0)\delta_{ij}. \tag{47}$$

Notice that in one dimension the initial state Eqs. (46) and (47) is translationally invariant and reads

$$\langle n^i_m(0)n^j_l(0)\rangle = \langle n^i_{r=|m-l|}(0)n^j_0(0)\rangle = \mathcal{K}_{ij}(1-\delta_{r,0})r^{-\Delta_{ij}} + \rho_i(0)\delta_{i,j}\delta_{r,0},\quad \mathcal{K}_{ij}=\mathcal{K}_{ji},\quad \triangle_{ij}=\triangle_{ji}. \tag{48}$$

This translational invariance which is broken [see Eq. (46)] in higher $(d\geqslant 2)$ dimensions leads to two regimes:

(i) We begin with the one-dimensional case $(d=1)$, here $r=r_\alpha\equiv|m-l|$. Because of the initial translational invariant state, we expect that the noninstantaneous correlation functions only depends on $r=m-l$, and we obtain

$$\langle n^A_m(t)n^A_l(0)\rangle = \langle n^A_r(t)n^A_0(0)\rangle = \rho_A(0)e^{-2\Gamma t}I_r(2\Gamma t) + \mathcal{K}_{AA}\sum_{r'\neq 0}|r'|^{-\Delta_{AA}}e^{-2\Gamma t}I_{r-r'}(2\Gamma t), \tag{49}$$

$$\langle n^A_m(t)n^B_l(0)\rangle = \mathcal{K}_{AB}\sum_{r'\neq 0}|r'|^{-\Delta_{AB}}e^{-2\Gamma t}I_{r-r'}(2\Gamma t) = \langle n^B_m(t)n^A_l(0)\rangle, \tag{50}$$

$$\langle n^B_m(t)n^B_l(0)\rangle = \langle n^B_r(t)n^B_0(0)\rangle = \rho_B(0)e^{-2\Gamma t}I_r(2\Gamma t) + \mathcal{K}_{BB}\sum_{r'\neq 0}|r'|^{-\Delta_{BB}}e^{-2\Gamma t}I_{r-r'}(2\Gamma t). \tag{51}$$

Because all the processes in the evolution operator are symmetric (unbiased), there is no drift, therefore $\langle n^{A,B}_r(t)n^{A,B}_0(0)\rangle = \langle n^{A,B}_{-r}(t)n^{A,B}_0(0)\rangle$.

(ii) In higher dimensions $(d\geqslant 2)$,

$$\langle n^A_{\mathbf{m}}(t)n^A_{\mathbf{l}}(0)\rangle = \rho_A(0)e^{-2d\Gamma t}\prod_{\alpha=1\dots d}I_{m_\alpha-m'_\alpha}(2\Gamma t) + \mathcal{K}_{AA}\sum_{(m'_1\neq l_1,\dots,m'_d\neq l_d)}\prod_{\alpha=1\dots d}\times|m'_\alpha-l_\alpha|^{-\Delta^\alpha_{AA}}e^{-2\Gamma t}I_{m_\alpha-m'_\alpha}(2\Gamma t), \tag{52}$$

$$\langle n^A_{\mathbf{m}}(t)n^B_{\mathbf{l}}(0)\rangle = \mathcal{K}_{AB}\sum_{(m'_1\neq l_1,\dots,m'_d\neq l_d)}\prod_{\alpha=1\dots d}\times|m'_\alpha-l_\alpha|^{-\Delta^\alpha_{AB}}e^{-2\Gamma t}I_{m_i-m'_i}(2\Gamma t), \tag{53}$$

$$\langle n^B_{\mathbf{m}}(t)n^B_{\mathbf{l}}(0)\rangle = \rho_B(0)e^{-2d\Gamma t}\prod_{\alpha=1\dots d}I_{m_\alpha-m'_\alpha}(2\Gamma t) + \mathcal{K}_{BB}\sum_{(m'_1\neq l_1,\dots,m'_d\neq l_d)}\prod_{\alpha=1\dots d}\times|m'_\alpha-l_\alpha|^{-\Delta^\alpha_{BB}}e^{-2\Gamma t}I_{m_\alpha-m'_\alpha}(2\Gamma t), \tag{54}$$

$$\langle n^B_{\mathbf{m}}(t)n^A_{\mathbf{l}}(0)\rangle = \mathcal{K}_{BA}\sum_{(m'_1\neq l_1,\dots,m'_d\neq l_d)}\prod_{\alpha=1\dots d}\times|m'_\alpha-l_\alpha|^{-\Delta^\alpha_{BA}}e^{-2\Gamma t}I_{m_i-m'_i}(2\Gamma t). \tag{55}$$

We observe that in higher dimensions, because of the initial correlations, the noninstantaneous correlation functions no longer depend on $|m_\alpha - l_\alpha|$.

We can express the asymptotic behavior of these noninstantaneous correlation functions in an unified way including both $d=1$ and $d\geqslant 2$ cases. Assuming that $r_\alpha=|m_\alpha-l_\alpha|\sim|m_\alpha|\gg 1$, with $r_\alpha=\sigma_\alpha L$ and $u=L^2/4\Gamma t<\infty$, $\Gamma t, r\gg 1$, the asymptotics read $[(i,j)\in(A,B)]$

$$\langle n_{\mathbf{m}}^{i}(t)n_{\mathbf{l}}^{j}(0)\rangle \sim \prod_{\alpha=1\dots d}\left[\frac{e^{-\sigma_{\alpha}^{2}u}}{1-\Delta_{ij}^{\alpha}}\sqrt{\frac{u\sigma_{\alpha}^{2}}{\pi}}\times\frac{1}{4(u\Gamma\sigma_{\alpha}^{2}t)^{\Delta_{ij}^{\alpha}/2}}\right]+\mathcal{O}(t^{-2d}),$$
$$0<\Delta_{ij}^{\alpha}<1. \tag{56}$$

Moreover,

$$\langle n_{\mathbf{m}}^{i}(t)n_{\mathbf{l}}^{j}(0)\rangle \sim \frac{1}{(4\pi\Gamma t)^{d/2}}\left(\rho_i(0)e^{-\sum_{\alpha=1}^{d}\sigma_{\alpha}^{2}u}\delta_{i,j}+\mathcal{K}_{ij}\prod_{\alpha=1\dots d}\zeta(\Delta_{ij}^{\alpha})+\mathcal{O}(t^{-2d})\right),\quad \Delta_{ij}^{\alpha}>1. \tag{57}$$

When $\Delta_{ij}^{\alpha}=1$, we have the marginal case with logarithmic corrections

$$\langle n_{\mathbf{m}}^{i}(t)n_{\mathbf{l}}^{j}(0)\rangle \sim \frac{1}{(4\pi\Gamma t)^{d/2}}\left(\rho_i(0)e^{-\sum_{\alpha=1}^{d}\sigma_{\alpha}^{2}u}\delta_{i,j}+\mathcal{K}_{ij}\prod_{\alpha=1\dots d}\ln(4u\sigma_{\alpha}\Gamma t)+\mathcal{O}(t^{-2d})\right),$$
$$\Delta_{ij}^{\alpha}=1. \tag{58}$$

Again, strong initial correlations lead to an algebraic decay of correlation functions $(0<\Delta_{ij}^{\alpha}<1)$, $\langle n_{\mathbf{m}}^{i}(t)n_{\mathbf{l}}^{j}(0)\rangle \sim 1/(4\pi\Gamma t)^{\Sigma_{\alpha}\Delta_{ij}^{\alpha}/2}$, $0<\Delta_{ij}^{\alpha}<1$, while for weak initial correlations, we have $\langle n_{\mathbf{m}}^{i}(t)n_{\mathbf{l}}^{j}(0)\rangle \sim 1/(4\pi\Gamma t)^{d/2}$, $\Delta_{ij}^{\alpha}>1$. The marginal case $\Delta_{ij}^{\alpha}=1$ is characterized by $\langle n_{\mathbf{m}}^{i}(t)n_{\mathbf{l}}^{j}(0)\rangle \sim (\ln 4\Gamma t)^{d}/(4\pi\Gamma t)^{d/2}$, $\Delta_{ij}^{\alpha}=1$.

## VII. SUMMARY AND CONCLUSION

In this work we studied, by analytical methods, the dynamics of a symmetric two-species reaction-diffusion model in arbitrary dimensions. We mapped this model onto an one-dimensional RSOS-type growth model and obtained explicit results for the latter. In particular, we were able to compute the density profile for three various initial conditions (uniform and nonuniform) in arbitrary dimensions.

Furthermore, we evaluated, for a translationally invariant system, the instantaneous two-point correlation functions in arbitrary dimensions. In one-spatial dimension, we considered the case where initial correlations were present. We observed that when the initial correlations are *strong enough*, they affect the asymptotic dynamics. We also noticed a crossover in the dynamics between the case of *strong* and *weak* initial correlations.

We mapped the one-dimensional version of the reaction-diffusion model on a ‘‘three-states’’ RSOS-type growth model. Using the exact, instantaneous correlation functions, we computed the exact expression of fluctuations of the interface for the latter model. We specifically considered the case of a ‘‘flat interface’’ where the fluctuations grow as $(\Gamma t)^{1/4}$, as in the corresponding ‘‘two-state’’ growth model; the three-state nature of the model considered only appears in the amplitude. We also considered the case where the initial configuration is translationally invariant, random and correlated. We saw that the initial correlations are ‘‘strong,’’ they affect the long-time behavior of the fluctuations of the displacement interface. Conversely, ‘‘weak’’ initial correlations, do not affect the dominant term and the fluctuations still grow as $(\Gamma t)^{1/4}$. This is analogous to what happens in the two-state RSOS systems where initial correlations affect the long-time behavior of the width [18].

Finally we computed in arbitrary dimensions the exact noninstantaneous two-point correlation functions for initially uncorrelated states as well as for cases where correlations were present. Here we again observed the effect of *strong* initial correlations on the dynamics and a crossover between regimes where *strong* and *weak* initial correlations takes place.

We conclude this work by addressing an interesting question based on the similarity of the *stochastic Hamiltonian* under consideration with the integrable Sutherland's quantum spin system [22,23]. It is known that for the one-dimensional SEP model, the relation with the Heisenberg chain has been fruitful to obtain a *(dynamical) matrix formulation* of the probability distribution, which allowed to solve the *(dynamical)* density profile for a SEP model with open boundary conditions (particles were injected and ejected from both ends of the chain) [8]. The relation of the present model to Sutherland's one suggests that a *dynamical matrix approach* would be possible (in one-spatial dimension) to treat the injection and ejection of particles at the boundary (open boundary conditions).

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## APPENDIX: SOLUTION OF THE EQUATIONS OF MOTION OF THE INSTANTANEOUS TWO-POINT CORRELATION FUNCTIONS

Solving the equations of motions (16) for the instantaneous correlation functions in arbitrary dimensions, for the model under consideration in Sec. VI, we obtain the following expression:

$$\mathcal{C}^{ij}_{|r|}(t)=\mathcal{C}^{ij}_{0}(0)e^{-4d\Gamma t}\prod_{\alpha=1}^{d} I_{r_\alpha}(4\Gamma t)+\sum_{r'\neq 0}\mathcal{C}^{ij}_{r'}(0)e^{-4\Gamma dt}\prod_{\alpha=1}^{d} I_{r_\alpha-r'_\alpha}(4\Gamma t)+4d\Gamma\mathcal{C}^{ij}_{0}(0)\int_0^t dt' e^{-4\Gamma d(t-t')}$$
$$\times\prod_{\alpha=1}^{d} I_{r_\alpha}[4\Gamma(t-t')]-2\Gamma\mathcal{C}^{ij}_{0}(0)\int_0^t dt' e^{-4\Gamma d(t-t')}$$
$$\times\sum_{\alpha=1}^{d}\left(\prod_{\alpha\neq\alpha'=1\dots d} I_{r_{\alpha'}}[4\Gamma(t-t')]\{I_{r_\alpha+1}[4\Gamma(t-t')]+I_{r_\alpha-1}[4\Gamma(t-t')]\}\right)+2\Gamma\int_0^t dt' e^{-4\Gamma d(t-t')}$$
$$\times\sum_{\alpha=1}^{d}\mathcal{C}^{ij}_{e^\alpha}(t')\left(\prod_{\alpha\neq\alpha'=1\dots d} I_{r_{\alpha'}}[4\Gamma(t-t')]\{I_{r_\alpha+1}[4\Gamma(t-t')]+I_{r_\alpha-1}[4\Gamma(t-t')]\}\right)$$
$$-4d\Gamma\int_0^t dt' e^{-4\Gamma d(t-t')}\mathcal{C}^{ij}_{e^\alpha}(t')\prod_{\alpha=1}^{d} I_{r_\alpha}[4\Gamma(t-t')]. \quad \text{(A1)}$$

Using the properties of the derivatives of Bessel functions and then integrating by parts, we obtain the more compact form (17).


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